

## Binary operations

### Examples:

The following are familiar binary operations:

1.)  $+$ ,  $-$  on  $\mathbb{R}$  (or  $\mathbb{Q}$  or  $\mathbb{Z}$ )

2.)  $\div$  on  $\mathbb{R} - \{0\}$  or  $\mathbb{Q} - \{0\}$  (why not  $\mathbb{Z} - \{0\}$ ?)

3.) Addition "mod 3". i.e.  $\{0, 1, 2\}$ , where  $a + b =$  remainder when dividing by 3.

So the addition table looks like

$+$	$0$	$1$	$2$
$0$	$0$	$1$	$2$
$1$	$1$	$2$	$0$
$2$	$2$	$0$	$1$

4.)  $\min$  is a binary operation on  $\mathbb{R}$  defined  $\min(a, b) = \begin{cases} a & \text{if } a \leq b \\ b & \text{if } b < a \end{cases}$

We can also come up w/ binary operations:

5.) Define  $*$  on  $\mathbb{R}$  to be  $a * b = 2a + 5b$ .

Def: If  $A$  is a set, then a binary operation  $*$  on  $A$  is a function

$$* : A \times A \rightarrow A$$

Denote  $*(a, b)$  by  $a * b$ .

Def: If  $B \subseteq A$  is a subset and  $*$  is a binary operation on  $A$ , then

$B$  is closed under  $*$  if for all  $x, y \in B$ ,  $x * y \in B$  as well.

In this case  $*$  restricted to  $B$  is the induced operation of  $*$  on  $B$ .

Ex:  $\cdot \mathbb{Z} \subseteq \mathbb{R}$  is closed under " $-$ "; but  $\mathbb{Z}_+ \subseteq \mathbb{R}$  is not closed under " $-$ " since  $1 - 1 \notin \mathbb{Z}_+$ .

$\cdot \mathbb{Q} - \{0\} \subseteq \mathbb{R} - \{0\}$  is closed under  $\div$ , but  $\mathbb{Z} - \{0\} \subseteq \mathbb{R} - \{0\}$  is not, since  $1 \div 2 = \frac{1}{2} \notin \mathbb{Z} - \{0\}$ .

Def:  $*$  on  $A$  is commutative if  $a * b = b * a \quad \forall a, b \in A$ .

Ex: Let  $A = \{ f : \mathbb{R} \rightarrow \mathbb{R} \}$  = the set of functions from  $\mathbb{R}$  to  $\mathbb{R}$ .

$+$ ,  $\cdot$ , and  $\circ$  are all binary operations on  $A$ .

$f + g$  is defined  $(f + g)(x) = f(x) + g(x)$

$f \cdot g$  is defined  $(f \cdot g)(x) = f(x) \cdot g(x)$

$f \circ g$  is defined  $(f \circ g)(x) = f(g(x))$

In this case,  $+$  and  $\cdot$  are both commutative, but if

$f(x) = x^2$  and  $g(x) = x + 1$ , then

$(f \circ g)(x) = (x + 1)^2$  and  $(g \circ f)(x) = x^2 + 1$ , so  $f \circ g \neq g \circ f$ ,

so  $\circ$  is not commutative.

Def:  $*$  is associative if  $\forall a, b, c \in A, (a * b) * c = a * (b * c)$ .

Ex:  $\cdot$  on  $\mathbb{R}$  is associative:  $(ab)c = a(bc)$ . However,

$\div$  on  $\mathbb{R} - \{0\}$  is not associative:  $(2/1)/2 = 2/2 = 1$

but  $2/(1/2) = 4$ .